CANONICAL COMPONENTS OF CHARACTER VARIETIES OF ARITHMETIC TWO-BRIDGE LINK COMPLEMENTS

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ABSTRACT. The desingularizations of the canonical components of $\mathrm{SL}_2(\mathbb{C})$-character varieties of arithmetic two-bridge link groups are determined.

0. INTRODUCTION

The $\mathrm{SL}_2(\mathbb{C})$-character variety of a hyperbolic 3-manifold is one of the central topics in the study of hyperbolic geometry. However little is known about the algebro-geometric properties of the character variety of a hyperbolic 3-manifold as an algebraic variety. In [11] the structure of the $\mathrm{SL}_2(\mathbb{C})$-character varieties of torus knot groups were explicitly determined. In [9] Macasieb, Petersen and van Luijk studied properties of the $\mathrm{SL}_2(\mathbb{C})$-character varieties of a certain family of two-bridge knots which contains the twist knots. In fact they showed that the canonical components of the $\mathrm{SL}_2(\mathbb{C})$-character varieties of the twist knots are hyperelliptic curves. In [7] Landes studied the canonical component of the Whitehead link complement.

The Whitehead link complement is one of the examples of arithmetic two-bridge links. In determining the canonical component of the character variety of the Whitehead link complement it was crucial that it can be considered as a (singular) conic bundle over the projective line $\mathbb{P}^1 := \mathbb{P}^1_{\mathbb{C}}$ in a specific projective space, which made it easy to obtain an explicit minimal model of the canonical component as an algebraic surface. It is already seen in other examples Landes computed that the canonical components of hyperbolic two-bridge links are not necessarily conic bundles over $\mathbb{P}^1$ in general.

It is known ([4]) that there are only finitely many arithmetic two-bridge links in the 3-sphere S^3. In fact, there are only 4 such links, the figure 8 knot $4_1 = (5/3)$, the Whitehead link $5^2_1 = (8/3)$, $6^2_2 = (10/3)$ and $6^2_3 = (12/5)$ in the Rolfsen's table. The canonical component of the character variety of the figure 8 knot complement is well known, which is an elliptic curve (for instance, see [8], Corollary 4.1). In this note we study the canonical components of the $\mathrm{SL}_2(\mathbb{C})$-character varieties of the other three arithmetic two-bridge links. (Unfortunately there was an error on the determination of a minimal model in the Whitehead link case in Landes' paper [7], more specifically the proof of Corollary 1 seems wrong, which was crucial for the determination of a minimal model in her paper. We will also recompute that in this note. Note that still the statement of Theorem 1 in her paper [7] is true.) We can see that those also are (singular) conic bundles over $\mathbb{P}^1$. Hence we can characterize their desingularizations by following the same method in [7]. The main result in this note is as follows:

Key words and phrases. $\mathrm{SL}_2(\mathbb{C})$-character variety, arithmetic 3-manifold, two-bridge link.

Theorem. *The desingularizations of the canonical components of the* $\mathrm{SL}_2(\mathbb{C})$*-character varieties of* 5_1^2, 6_2^2 *and* 6_3^2 *are conic bundles over the projective line* $\mathbb{P}^1$ *which are isomorphic to the surface obtained from* $\mathbb{P}^1 \times \mathbb{P}^1$ *by repeating a one-point blow up* 9, 12 *and* 9 *times (or equivalently obtained from* $\mathbb{P}^2$ *by repeating a one-point blow up* 10, 13 *and* 10 *times), respectively.*

Here we explain the outline of this note. In Section 1 we will show the explicit defining equations of the natural models ($\mathrm{SL}_2(\mathbb{C})$-character varieties) of the arithmetic two-bridge links and study their irreducible components. In particular we identify their canonical components. In Section 2 we describe the singular points of certain projective models of the canonical components of the natural models which are equipped with the conic bundle structure over $\mathbb{P}^1$. We also compute explicitly the degenerate fibers of them, which is useful for the determination of minimal models of the desingularizations of those projective models. In Section 3 we determine minimal models of the desingularization of our projective models by employing intersection theory of surfaces. In Section 4 we characterize the desingularizations in terms of the number of blow ups from the minimal models obtained in Section 3 by computing the Euler characteristics of the projective models.

The author is grateful to Kenji Hashimoto for his help on the determination of minimal models of the rational surfaces appeared as the canonical components of the $\mathrm{SL}_2(\mathbb{C})$-character varieties of the arithmetic two-bridge links in this note. He would also like to express his gratitude to the referee for his/her comments on the original manuscript, which drastically improved the composition of this note.

1. Natural models

The $\mathrm{SL}_2(\mathbb{C})$-character variety of a manifold M is the set of characters of $\mathrm{SL}_2(\mathbb{C})$-representations of the fundamental group $\pi_1(M)$ of M, which is known to be an affine algebraic set. For basics and applications of $\mathrm{SL}_2(\mathbb{C})$-character varieties, see Culler and Shalen's original paper [3] or Shalen's survey paper [13]. It is known in general that we can compute the defining polynomials of $\mathrm{SL}_2(\mathbb{C})$-character varieties of finitely generated groups explicitly from their group presentations ([5], Theorem 3.2). However in this note we only consider two-bridge link groups. In this case we can compute the defining polynomials by Riley's method ([12], §2 or see [7], §4) by which we can compute the defining polynomials with less computation.

Here we only show the result of computation of the defining polynomials of the $\mathrm{SL}_2(\mathbb{C})$-character varieties for the arithmetic two-bridge link groups and show which irreducible component is the canonical component (that is, the irreducible component containing the point corresponding to the holonomy representation). We also include the Whitehead link case for the convenience of the reader. For the detailed way of the computation of the defining polynomials, see [7], §4.

1.1. **Preliminary:Notation.** Here we summarize some basic results on group presentations of the fundamental group of two-bridge link groups.

Let L be a two-bridge link in the 3-sphere. Then it is well-known (cf. [2], Chapter 12, G, E 12.1) that its fundamental group has the following group presentation:

$$\pi_1(S^3 \setminus L) \xrightarrow{\sim} \langle\, a, b \mid awAW = 1 \rangle,$$

where A and B mean the inverses a^{-1} and b^{-1}, respectively. When L is represented by the Schubert's normal form (α/β), the word w is defined by

$$w := b^{\epsilon_1} a^{\epsilon_2} b^{\epsilon_3} \cdots a^{\epsilon_{\alpha-2}} b^{\epsilon_{\alpha-1}},$$

where $\epsilon_i := (-1)^{\left[\frac{i\beta}{\alpha}\right]}$. Here, for a real number r, $[r]$ is the maximal integer not greater than r.

The $\mathrm{SL}_2(\mathbb{C})$-character variety $X(M)$ of a manifold M is the set of $\mathrm{SL}_2(\mathbb{C})$-characters of $\pi_1(M)$, i.e., $X(M) := \left\{\chi_\rho := \mathrm{Tr}(\rho) : \pi_1(M) \to \mathbb{C} \mid \rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})\right\}$.

If M is an orientable complete hyperbolic 3-manifold, there is a special irreducible component containing the point corresponding to the character of the holonomy representation of M. It is known (cf. [13], Theorem 4.5.1) that the canonical component of the $\mathrm{SL}_2(\mathbb{C})$-character variety of an n-component hyperbolic link complement has dimension n. Especially, the canonical component of the $\mathrm{SL}_2(\mathbb{C})$-character variety of a hyperbolic two-bridge link complement is an irreducible affine surface over $\mathbb{C}$.

Any $\mathrm{SL}_2(\mathbb{C})$-character χ of $\pi_1(S^3 \setminus L)$ is determined by the values $\chi(a), \chi(b), \chi(ab)$. Thus we have a canonical injection $X(S^3 \setminus L) \to \mathbb{A}^3 := \mathbb{C}^3$ defined by

$$\chi_\rho \mapsto (x, y, z) := (\chi_\rho(a), \chi_\rho(b), \chi_\rho(ab)).$$

Put $q := x^2 + y^2 + z^2 - xyz - 4$. It is known (cf. [10], Lemma 1.2.3) that a representation $\pi_1(S^3 \setminus L) \to \mathrm{SL}_2(\mathbb{C})$ is reducible if and only if $q(x, y, z) = 0$. In particular the points corresponding to abelian characters are contained in the algebraic set $V(q)$. Here, for polynomials $f_1, \cdots, f_r \in \mathbb{C}[x, y, z]$ let $V(f_1, \cdots, f_r)$ be the set of common zeros of $f_1, \cdots, f_r$, i.e.

$$V(f_1, \cdots, f_r) := \{(x, y, z) \in \mathbb{A}^3 \mid f_i(x, y, z) = 0,\ 1 \le i \le r\}.$$

1.2. **Whitehead link 5_1^2 case.** The fundamental group $\pi_1(S^3 \setminus 5_1^2)$ of the Whitehead link complement $5_1^2 = (8/3)$ has $w := baBABab$. Then $\mathrm{SL}_2(\mathbb{C})$-character variety of $S^3 \setminus 5_1^2$ is defined by the following two polynomials

$$f_0 := p_0 q, \quad g_0 := p_0(y-2)(y+2),$$

where $p_0 := z^3 - xyz^2 + (x^2 + y^2 - 2)z - xy$ and q are irreducible in $\mathbb{C}[x, y, z]$ (for instance, if we assume there is a factorization of p_0, immediately we have a contradiction by comparing degrees of monomials on both sides. We can show that q is irreducible in the same manner). Hence we have the decomposition

$$X(S^3 \setminus 5_1^2) = V(p_0) \cup V(q, y-2) \cup V(q, y+2).$$

Here $V(q, y-2)$ and $V(q, y+2)$ are affine lines $\mathbb{A}^1$. Now the affine algebraic set $V(p_0)$ defined by p_0 is the unique 2-dimensional component of the character variety of 5_1^2. Hence that is the canonical component $X_0(S^3 \setminus 5_1^2)$ of 5_1^2. The points of the algebraic set defined by the polynomial q correspond to the reducible representations of $\pi_1(S^3 \setminus 5_1^2)$. We summerize that the natural model $X(S^3 \setminus 5_1^2)$ consists of three irreducible algebraic sets $V(p_0), V(q, y-2)$ and $V(q, y+2)$. The canonical component $X_0(S^3 \setminus 5_1^2) = V(p_0)$ is the unique irreducible algebraic subset of $X(S^3 \setminus 5_1^2)$ of dimension 2. The other two components consist of points corresponding to the reducible $\mathrm{SL}_2(\mathbb{C})$-characters of $\pi_1(S^3 \setminus 5_1^2)$.

1.3. 6_2^2 **case.** The fundamental group of the arithmetic two-bridge link $6_2^2 = (10/3)$ has $w := babABAbab$. Then the $\mathrm{SL}_2(\mathbb{C})$-character variety of 6_2^2 is defined by the following two polynomials

$$f_1 := p_1 q, \quad g_1 := p_1(y-2)(y+2),$$

where $p_1 := z^4 - xyz^3 + (x^2+y^2-3)z^2 - xyz + 1$. Note that p_1 is irreducible in $\mathbb{C}[x,y,z]$ by the similar argument as in the Whitehead link case. The $\mathrm{SL}_2(\mathbb{C})$-character variety $X(S^3 \setminus 6_2^2)$ consists of three algebraic sets

$$X(S^3 \setminus 6_2^2) = V(p_1) \cup V(q, y-2) \cup V(q, y+2).$$

Here $V(q, y-2)$ and $V(q, y+2)$ are affine lines $\mathbb{A}^1$. Now the affine algebraic set $V(p_1)$ defined by p_1 is the unique 2-dimensional component of the character variety of 6_2^2. Hence that is the canonical component of 6_2^2.

Thus the natural model $X(S^3 \setminus 6_2^2)$ consists of three irreducible components, the canonical component $X_0(S^3 \setminus 6_2^2) = V(p_1)$ and two components $V(q, y-2)$ and $V(q, y+2)$ which correspond to $\mathrm{SL}_2(\mathbb{C})$-reducible characters of $\pi_1(S^3 \setminus 6_2^2)$.

1.4. 6_3^2 **case.** The fundamental group of the arithmetic two-bridge link $6_3^2 = (12/5)$ has $w := baBAbabABab$. Then $\mathrm{SL}_2(\mathbb{C})$-character variety of 6_3^2 is defined by the following two polynomials

$$f_2 := p_2 q r, \quad g_2 := p_2 r(y-2)(y+2),$$

where $p_2 := z^3 - xyz^2 + (x^2+y^2-1)z - xy$ and $r := x^2+y^2+z^2-xyz-3$ are irreducible polynomials in $\mathbb{C}[x,y,z]$ by the similar argument as in the Whitehead link case. Thus we have the decomposition

$$X(S^3 \setminus 6_3^2) = V(p_2) \cup V(r) \cup V(q, (y-2)(y+2)).$$

Here $V(q, (y-2)(y+2))$ is the union of two affine lines $\mathbb{A}^1$. There are two irreducible components of dimension 2 in this case. By considering the hyperbolicity equations of $S^3 \setminus 6_3^2$ and the fact that the images of meridians of holonomy representations are parabolic elements, we can compute the holonomy representation concretely. In fact, the holonomy representation of $S^3 \setminus 6_3^2$ is defined by

$$a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \text{ where } \alpha = \frac{-1+\sqrt{-7}}{2}$$

up to conjugation. Then the point corresponding to the holonomy representation is $(x,y,z) = (2,2,2+\alpha)$, which is a zero of the polynomial p_2. Hence the canonical component of 6_3^2 is the irreducible component $V(p_2)$. The other irreducible component of dimension 2, $V(r)$ is a smooth affine cubic surface. Moreover, we see that its natural homogenization $V_+(R) \subset \mathbb{P}^3$ defined by $R := (x^2+y^2+z^2)w + xyz - 3w^3$ is a smooth projective cubic surface. A cubic surface in $\mathbb{P}^3$ is a well-studied object. It is a Del Pezzo surface of degree 3, which is isomorphic to $\mathbb{P}^2$ with six points blown up (or $\mathbb{P}^1 \times \mathbb{P}^1$ with five points blown up since $\mathbb{P}^2$ with two points blown up is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with one point blown up, cf. [6], V, Remark 4.7.1).

Thus the natural model $X(S^3 \setminus 6_3^2)$ consists of four irreducible components, two of which are the canonical component $X_0(S^3 \setminus 6_3^2) = V(p_2)$ and a smooth affine surface $V(r)$. The other two components correspond to reducible $\mathrm{SL}_2(\mathbb{C})$-characters of $\pi_1(S^3 \setminus 6_3^2)$.

2. Projective models

Let p_0, p_1 and p_2 be the polynomials in the previous section which define the canonical component of the $\mathrm{SL}_2(\mathbb{C})$-character variety of the Whitehead link, 6_2^2 and 6_3^2 link respectively. Then the Jacobian criterion shows that $V(p_i)$ is a smooth affine surface for any i. The projective surfaces in $\mathbb{P}^3$ obtained by homogenizing p_i naturally have infinitely many singularities. Thus we consider another compactification, namely a compactification in $\mathbb{P}^2 \times \mathbb{P}^1$ to obtain a projective surface having less singularities. We follow the method introduced in [7] and [9]. After reviewing A_n-singularities in Subsection 2.1 we study the homogenizations S_i of $V(p_i)$ in $\mathbb{P}^2 \times \mathbb{P}^1$.

2.1. **A_n-singularities.** The du Val singularity (or rational double points) is one kind of isolated singularity of a complex surface whose exceptional curve consists of a tree of rational smooth curves, which is the unique rational singularity for hypersurfaces in $\mathbb{A}^3$. It is classified into three types (A-D-E singularities). Here we only explain the A_n-singularity. The A_n-singularity is one type of the du Val singularity characterized by the singular point $(0,0,0)$ of the equation $x^2 + y^2 + z^{n+1} = 0$. The exceptional curve of $x^2 + y^2 + z^{n+1} = 0$ at the singular point $(0,0,0)$ obtained by blowing up some number of times consists of n smooth projective irreducible curves (isomorphic to $\mathbb{P}^1$) with self-intersection number -2, which intersects each other transversally described as Figure 1.

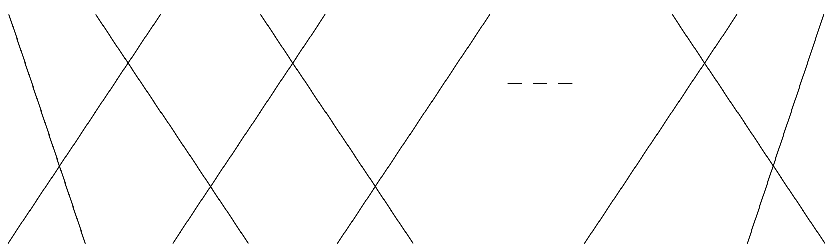


Figure 1. Exceptional curve of A_n-singularity

Each curve on both sides meets only with another curve at one point. The other curves meet with two other curves transversally. It is also expressed by the Dynkin diagram

$$\circ \text{———} \circ \text{———} \circ - - - \circ \text{———} \circ$$

Figure 2. Dynkin diagram of A_n-singularity

We have a relation between the topological Euler characteristics of a (singular) surface and its desingularization as follows.

Lemma 2.1 (cf. [7], Prop. 2)**.** *Let S be an irreducible smooth projective surface over $\mathbb{C}$ and p a point of S. Let $\tilde{S}$ be the blow up of S at p. Then $\chi(\tilde{S}) = \chi(S) + 1$.*

Lemma 2.2. *Let S be an irreducible projective surface over $\mathbb{C}$, $p \in S$ an A_n-singular point and let $\epsilon : \tilde{S} \to S$ be the desingularization of S at the point p. Then we have $\chi(\tilde{S}) = \chi(S) + n$.*

Proof. Note that the fiber of $\epsilon : \tilde{S} \to S$ at the point p consists of n projective lines which intersects with each other as in Figure 1. Since $\chi(\mathbb{P}^1) = 2$ and $\chi(\text{point}) = 1$, we have

$$\begin{aligned}\chi(\tilde{S}) &= \chi(\tilde{S} \setminus \epsilon^{-1}(p)) + \chi(\epsilon^{-1}(p)) = \chi(S \setminus \{p\}) + (n\chi(\mathbb{P}^1) - (n-1)) \\ &= \chi(S) - 1 + n + 1 = \chi(S) + n.\end{aligned}$$

□

2.2. **Projective models of the canonical components.** Let

$$\mathbb{P}^2 \times \mathbb{P}^1 := \{(x : y : u,\ z : w) \mid (x : y : u) \in \mathbb{P}^2,\ (z : w) \in \mathbb{P}^1\}$$

be the product of $\mathbb{P}^2$ and $\mathbb{P}^1$, and let

$$\begin{aligned}F_0 &:= u^2z^3 - xyz^2w + (x^2 + y^2 - 2u^2)zw^2 - xyw^3 \\ F_1 &:= u^2z^4 - xyz^3w + (x^2 + y^2 - 3u^2)z^2w^2 - xyzw^3 + u^2w^4 \\ F_2 &:= u^2z^3 - xyz^2w + (x^2 + y^2 - u^2)zw^2 - xyw^3\end{aligned}$$

be the homogenization of p_0, p_1 and p_2 in $\mathbb{P}^2 \times \mathbb{P}^1$. Consider the algebraic set

$$S_i := V(F_i) := \{(x : y : u,\ z : w) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid F_i(x, y, u, z, w) = 0\}$$

defined by F_i in $\mathbb{P}^2 \times \mathbb{P}^1$. Since $\mathbb{A}^3$ is naturally embedded in $\mathbb{P}^2 \times \mathbb{P}^1$ as

$$\{(x : y : 1,\ z : 1) \mid (x, y) \in \mathbb{A}^2,\ z \in \mathbb{A}^1\},$$

$V(p_i)$ is embedded in S_i birationally.

Let $\phi : \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection which is defined by $(x : y : u,\ z : w) \mapsto (z : w)$ and define ϕ_i by the restriction of ϕ on S_i. We note that all the fibers of S_i except finitely many points are smooth conic in $\mathbb{P}^2$. Hence ϕ_i defines a (singular) conic bundle structure on S_i. In the following subsections we show the explicit description of the singular (degenerate) fibers of ϕ_i and compute the Euler characteristic $\chi(\tilde{S}_i)$ in terms of $\chi(S_i)$.

2.3. **Whitehead link case.** Let $F_0 := u^2z^3 - xyz^2w + (x^2 + y^2 - 2u^2)zw^2 - xyw^3$ be the homogenization of the polynomial $p_0 = z^3 - xyz^2 + (x^2 + y^2 - 2)z - xy$ in the projective space $\mathbb{P}^2 \times \mathbb{P}^1$. Let $S_0 := V(F_0)$ be the algebraic set defined by F_0 in $\mathbb{P}^2 \times \mathbb{P}^1$. It is shown in [7], §4 that S has only four singular points

$$(1 : 0 : 0,\ 1 : 0),\ (0 : 1 : 0,\ 1 : 0),\ (1 : 1 : 0,\ 1 : 1),\ (1 : -1 : 0,\ 1 : -1).$$

These four points are A_1 singularities (we can resolve the singularity by blowing up once) and the exceptional curves at the singular points are isomorphic to the projective line $\mathbb{P}^1$. Thus we have the following relation on the topological Euler characteristic of S_0 and the desingularization $\tilde{S_0}$ by Lemma 2.2:

$$\chi(\tilde{S_0}) = \chi(S_0 \setminus S_{0,\text{sing}}) + 4\chi(\mathbb{P}^1) = \chi(S_0) - 4 + 4\chi(\mathbb{P}^1) = \chi(S_0) + 4.$$

Here $S_{0,\text{sing}}$ is the set of singular points of S_0. In Section 4 we compute the topological Euler characteristic $\chi(S_0)$ and determine $\tilde{S}_0$ in terms of the number of one-point blow ups from a minimal model of $\tilde{S}_0$.

Note that we can consider the surface S_0 (hence $\tilde{S}_0$) as a (singular) conic bundle over $\mathbb{P}^1$ by the projection $\phi_0 : S_0 \to \mathbb{P}^1$ which is defined by $(x : y : u,\ z : w) \mapsto (z : w)$. It has six

degenerate fibers at $(1:0)$, $(0:1)$, $(1:\pm 1)$, $(1:\pm\frac{1}{\sqrt{2}})$. In fact, the degenerate fibers of $\phi_0 : S_0 \to \mathbb{P}^1$ are expressed as follows:

$$\begin{aligned}
\phi_0^{-1}(1:0) &\tilde{\to} \{(x:y:u) \in \mathbb{P}^2 \mid u^2 = 0\},\\
\phi_0^{-1}(0:1) &\tilde{\to} \{(x:y:u) \in \mathbb{P}^2 \mid xy = 0\},\\
\phi_0^{-1}(1:\pm\tfrac{1}{\sqrt{2}}) &\tilde{\to} \{(x:y:u) \in \mathbb{P}^2 \mid \tfrac{1}{2}(x \mp \sqrt{2}y)(x \mp \tfrac{1}{\sqrt{2}}y) = 0\},\\
\phi_0^{-1}(1:\pm 1) &\tilde{\to} \{(x:y:u) \in \mathbb{P}^2 \mid (x \mp y) - u)((x \mp y) + u) = 0\}.
\end{aligned}$$

Note that the fiber $\phi_0^{-1}(1:0)$ contains the singular points $(1:0:0,\ 1:0)$, $(0:1:0,\ 1:0)$ of the surface S_0. The fiber $\phi_0^{-1}(1:\pm 1)$ contains the singular point $(1:\pm 1:0,\ 1:\pm 1)$ of the surface S_0 respectively.

2.4. 6_2^2 **case.** Let $F_1 := u^2z^4 - xyz^3w + (x^2+y^2-3u^2)z^2w^2 - xyzw^3 + u^2w^4$ be the homogenization of $p_1 = z^4 - xyz^3 + (x^2+y^2-3)z^2 - xyz + 1$ in $\mathbb{P}^2 \times \mathbb{P}^1$ with coordinates x, y, u and z, w, and let

$$S_1 := V(F_1) := \{(x:y:u,\ z:w) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid F_1(x,y,u,z,w) = 0\}$$

be the algebraic set defined by F_1 in $\mathbb{P}^2 \times \mathbb{P}^1$. (This is symmetric on x, y and z, w.) This projective surface has only finitely many singular points. In fact, its singularities are only the following six points:

$$\begin{aligned}
&(1:0:0,\ 1:0),\ (0:1:0,\ 1:0),\ (1:0:0,\ 0:1),\\
&(0:1:0,\ 0:1),\ (1:1:0,\ 1:1),\ (1:-1:0,\ 1:-1),
\end{aligned}$$

which are A_1 singularities. Especially they are resolved by one blow up at each point. Let $\tilde{S}_1 \to S_1$ be the desingularization of S_1 blown up at these six points. Then the exceptional curve at each singular point is isomorphic to $\mathbb{P}^1$. Thus we have the following relation on the topological Euler characteristic of S_1 and $\tilde{S}_1$ by Lemma 2.2:

$$\chi(\tilde{S}_1) = \chi(S_1 \setminus S_{1,\mathrm{sing}}) + 6\chi(\mathbb{P}^1) = \chi(S_1) - 6 + 6\chi(\mathbb{P}^1) = \chi(S_1) + 6.$$

As a (singular) conic bundle over $\mathbb{P}^1$ by the projection $\phi_1 : S_1 \to \mathbb{P}^1$ the surface S_1 (hence $\tilde{S}_1$) has eight degenerate fibers at $(1:0)$, $(0:1)$, $(1:\pm 1)$, $(1:\pm\frac{\sqrt{5}\pm 1}{2})$. In fact, they are written as follows:

$$\begin{aligned}
\phi_1^{-1}(1:0) &\tilde{\to} \{(x:y:u) \in \mathbb{P}^2 \mid u^2 = 0\},\\
\phi_1^{-1}(0:1) &\tilde{\to} \{(x:y:u) \in \mathbb{P}^2 \mid u^2 = 0\},\\
\phi_1^{-1}(1:\tfrac{\sqrt{5}\pm 1}{2}) &\tilde{\to} \{(x:y:u) \in \mathbb{P}^2 \mid \tfrac{3\pm\sqrt{5}}{2}(x - \tfrac{\sqrt{5}+1}{2}y)(x - \tfrac{\sqrt{5}-1}{2}y) = 0\},\\
\phi_1^{-1}(1:-\tfrac{\sqrt{5}\pm 1}{2}) &\tilde{\to} \{(x:y:u) \in \mathbb{P}^2 \mid \tfrac{3\pm\sqrt{5}}{2}(x + \tfrac{\sqrt{5}+1}{2}y)(x + \tfrac{\sqrt{5}-1}{2}y) = 0\},\\
\phi_1^{-1}(1:\pm 1) &\tilde{\to} \{(x:y:u) \in \mathbb{P}^2 \mid ((x \mp y) - u)((x \mp y) + u) = 0\}.
\end{aligned}$$

We remark that the fiber $\phi_1^{-1}(1:0)$ (resp. $\phi_1^{-1}(0:1)$) contains the singular points $(1:0:0,\ 1:0)$, $(0:1:0,\ 1:0)$ (resp. $(1:0:0,\ 0:1)$, $(0:1:0,\ 0:1)$) of S_1, and that each fiber $\phi_1^{-1}(1:\pm 1)$ contains the singular point $(1:\pm 1:0,\ 1:\pm 1)$ of S_1. In the next section we compute a minimal model of the desingularization of the surface S_1.

2.5. 6_3^2 **case.** Let $p_2 := z^3 - xyz^2 + (x^2+y^2-1)z - xy$. Let $F_2 = u^2z^3 - xyz^2w + (x^2+y^2-u^2)zw^2 - xyw^3$ be the homogenization of p_2 in $\mathbb{P}^2\times\mathbb{P}^1$ with the coordinates $(x:y:u,\ z:w)$. The corresponding projective surface $S_2 = V(F_2) \subset \mathbb{P}^2\times\mathbb{P}^1$ has the following four singular points:

$$(1:0:0,\ 1:0),\ (0:1:0,\ 1:0),\ (1:1:0,\ 1:1),\ (1:-1:0,\ 1:-1).$$

We remark that the first two points are A_1 singularities and the other two points are A_3 singularities. Hence we can resolve the first two singularities by blowing up once at each point but we have to blow up twice for the latter two points. Let $\tilde{S}_2$ be the smooth projective surface obtained by blowing up S_2 at these four singular points. The exceptional curves at the singular points $(1:0:0,\ 1:0)$ and $(0:1:0,\ 1:0)$ are isomorphic to $\mathbb{P}^1$, and the exceptional curve at $(1:1:0,\ 1:1)$ (resp. $(1:-1:0,\ 1:-1)$) is the union of three curves E_1^+, E_2^+ and E_3^+ (resp. E_1^-, E_2^- and E_3^-) respectively. Here $E_i^\pm$ are smooth projective curves isomorphic to $\mathbb{P}^1$ with self-intersection number -2. The curve $E_2^\pm$ intersects with $E_1^\pm$ and $E_3^\pm$ at one point respectively and $E_1^\pm$ and $E_3^\pm$ do not intersect each other (see Figure 1). Thus we can express the Euler characteristic of $\tilde{S}_2$ in terms of that of S_2 by Lemma 2.2:

$$\chi(\tilde{S}_2) = \chi(S_2 \smallsetminus S_{2,\mathrm{sing}}) + 2\chi(\mathbb{P}^1) + 2\chi(E_1^+ \vee E_2^+ \vee E_3^+) = \chi(S_2) - 4 + 4 + 8 = \chi(S_2) + 8.$$

We can consider the surface S_2 (hence $\tilde{S}_2$) as a conic bundle over $\mathbb{P}^1$ by the projection $\phi_2 : S_2 \to \mathbb{P}^1$. It has four degenerate fibers at $(1:0)$, $(0:1)$, $(1:\pm1)$. In fact, the degenerate fibers of $\phi_2 : S_2 \to \mathbb{P}^1$ are expressed as follows:

$$\begin{aligned}
\phi_2^{-1}(1:0) &\tilde{\to} \{(x:y:u) \in \mathbb{P}^2 \mid u^2 = 0\},\\
\phi_2^{-1}(0:1) &\tilde{\to} \{(x:y:u) \in \mathbb{P}^2 \mid xy = 0\},\\
\phi_2^{-1}(1:\pm1) &\tilde{\to} \{(x:y:u) \in \mathbb{P}^2 \mid (x \mp y)^2 = 0\}.
\end{aligned}$$

We note that the fiber $\phi_2^{-1}(1:0)$ contains the singular points $(1:0:0,\ 1:0)$, $(0:1:0,\ 1:0)$ of S_2. The fiber $\phi_2^{-1}(1:\pm1)$ contains the singular point $(1:\pm1:0,\ 1:\pm1)$ of the surface S_2 respectively.

3. Minimal models

Since all the three surfaces S_0, S_1, S_2 are rational surfaces, their minimal models are either the projective plane $\mathbb{P}^2$ or the Hirzebruch surfaces $\mathbb{F}_n$ ($n \geq 0$, $n \neq 1$) (cf. [1], Theorem V.10). Here we compute a minimal model of the surface S_i for each i, which is obtained naturally from its fibered surface structure. The purpose of this section is to prove the following two lemmas.

Lemma 3.1. *For each $\tilde{S}_i$ we can blow down $\tilde{S}_i$ over $\mathbb{P}^1$ some number of times so that it becomes a geometrically ruled surface T_i over $\mathbb{P}^1$, namely all the fibers are isomorphic to $\mathbb{P}^1$.*

Lemma 3.2. *T_i is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.*

In Subsection 3.1 we review some terminology on algebraic surfaces and some basic facts on the intersection theory of algebraic surfaces and minimal models. In Subsection 3.2 and Subsection 3.3 we show Lemma 3.1 and Lemma 3.2.

3.1. **Preliminary on algebraic surfaces.**

3.1.1. *Basic properties of intersection theory of surfaces.* Here we summarize some basic properties of the intersection theory of algebraic surfaces. In particular we include some results on the intersection numbers of divisors of fibered surfaces, which are necessary for the computation of minimal models obtained by blowing down the conic bundles over $\mathbb{P}^1$ appeared in the previous section. For more details, see [1], [6] or [14], III §7, 8.

In Subsubsection 3.1.1 and Subsubsection 3.1.2, a curve or a surface always means a smooth projective irreducible curve or surface unless otherwise mentioned.

Let S be a surface over the field of complex numbers $\mathbb{C}$. Let $\mathrm{Div}(S)$ be the group of all the divisors of S and $\mathrm{div} : \mathbb{C}(S)^{\times} \to \mathrm{Div}(S)$ the divisor function. We say that two divisors $D, D' \in \mathrm{Div}(S)$ are *linearly equivalent* if $D' = D + \mathrm{div}(f)$ for some $f \in \mathbb{C}(S)^{\times}$. Then there is a unique symmetric bilinear pairing

$$(\ , \ ) : \mathrm{Div}(S) \times \mathrm{Div}(S) \to \mathbb{Z}$$

which satisfies the following two properties:

(1) For curves C_1, C_2 on S which meet everywhere transversally, then $(C_1, C_2) = \#(C_1 \cap C_2)$, where $\#(C_1 \cap C_2)$ is the number of points of $C_1 \cap C_2$.
(2) If $D, D_1, D_2 \in \mathrm{Div}(S)$ are divisors and D_1, D_2 are linearly equivalent, then $(D, D_1) = (D, D_2)$. Hence the pairing $(\ , \ )$ induces the pairing $\mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathbb{Z}$ on the Picard group of S (the quotient group of $\mathrm{Div}(S)$ by the image of div).

We will also write $D \cdot D'$ in place of (D, D') for two divisors $D, D' \in \mathrm{Div}(S)$. For any divisor $D \in \mathrm{Div}(S)$ we call $D^2 := D \cdot D$ the *self-intersection number* of D.

We say that a surface S is a *fibered surface* over a curve C if there is a surjective morphism $\pi : S \to C$. The fiber $\pi^{-1}(t)$ for a point $t \in C$ is a smooth projective curve for all but finitely many points of C. For any curve D on a fibered surface S over C its image $\pi(D)$ is either a point or C. A curve D on S is called *fibral* (or *vertical*) if the image is a point, and is called *horizontal* if the image is the curve C. A divisor $D \in \mathrm{Div}(S)$ is called vertical (horizontal) if all the curves appeared as the components of D are vertical (horizontal). The structure morphism $\pi : S \to C$ of a fibered surface induces the homomorphism

$$\pi^* : \mathrm{Div}(C) \to \mathrm{Div}(S)$$

defined by

$$\sum_{t \in C} n_t[t] \mapsto \sum_{t \in C} n_t \sum_{\Gamma \subset \pi^{-1}(t)} \mathrm{ord}_{\Gamma}(u_t \circ \pi)[\Gamma],$$

where u_t is a uniformizer of the function field $k(C)$ at t and Γ runs through all the curves in $\pi^{-1}(t)$, and $\mathrm{ord}_{\Gamma}(u_t \circ \pi)$ is the order of $u_t \circ \pi$ in the function field $k(S)$ of S by the discrete valuation ord_{Γ} defined by Γ. (For the definition of the inverse image of a divisor in general, see, e.g. [1], I or [6], II.6.)

Proposition 3.3 (cf. [1], Prop. I. 8)**.** (1) *Let $\pi : S \to C$ be a fibered surface over a curve C. If F is a fiber of ϕ (that is, $F = \pi^*[t]$ for some $t \in C$), then $F^2 = 0$.*
(2) *Let S, S' be surfaces and $g : S' \to S$ a generically finite morphism of degree d. If D, D' are divisors on S, then $g^*D \cdot g^*D' = d(D \cdot D')$.*

Lemma 3.4 ([14], Chapter III, Lemma 8.1)**.** *Let $\pi : S \to C$ be a fibered surface and $\delta \in \mathrm{Div}(C)$. If $D \in \mathrm{Div}(S)$ is a vertical divisor then $D \cdot \pi^*(\delta) = 0$.*

3.1.2. *Minimal models of surfaces.* Here we summarize some basic facts on minimal models we need in this note.

Let S be a surface over $\mathbb{C}$ and E a curve on S. A curve E on S is called an *exceptional curve* if it is obtained as a component of the fiber of a point by blowing up a (possibly singular) surface.

Proposition 3.5 (cf. [1], II, 1)**.** *Let S be a smooth surface and p a point on S. Let $\epsilon : \tilde{S} \to S$ be the blow up morphism of S at the point p. Then $\epsilon^{-1}(p)$ is an irreducible curve on S isomorphic to $\mathbb{P}^1$.*

We say that a curve E is a (-1)-*curve* if it is isomorphic to $\mathbb{P}^1$ and its self-intersection number is -1, that is, $E^2 = -1$.

Proposition 3.6 (cf. [1], Lem. II, 2, Prop. II, 3, (i), (ii))**.** *Let S be a surface and p a point on S. Let $\epsilon : \tilde{S} \to S$ be the blow up at p and E the exceptional curve of p.*

1. *There is an isomorphism $\mathrm{Pic}\, S \oplus \mathbb{Z} \xrightarrow{\sim} \mathrm{Pic}\, \tilde{S}$ defined by $(D, n) \mapsto \epsilon^* D + nE$.*
2. *If $D, D' \in \mathrm{Div}\, S$, then $\epsilon^* D \cdot \epsilon^* D' = D \cdot D'$, $E \cdot \epsilon^* D = 0$ and $E^2 = -1$.*
3. *If C is an irreducible projective curve on S which passes through the point p with multiplicity m, then $\epsilon^* C = \tilde{C} + mE$ ($\tilde{C}$ is the strict transform of C, namely the closure of $\epsilon^{-1}(C \setminus p)$ in $\tilde{S}$).*

Theorem 3.7 (Castelnuovo's contractibility Theorem, cf. [1], II, Theorem 17))**.** *Let S be a surface over $\mathbb{C}$ and E a (-1)-curve on S. Then E is an exceptional curve on S, namely there is a surface S' and a morphism $\epsilon : S \to S'$ such that ϵ is the blow up of S' at a point p with $E = \epsilon^{-1}(p)$.*

A surface S is called *relatively minimal* if there is no (-1)-curves on S. There are only finitely many (-1)-curves on a surface. Therefore, for a surface S we can find a sequence of surfaces

$$S \to S_1 \to S_2 \to \cdots \to S_n$$

such that S_n is relatively minimal. Such a surface S_n is called a *relatively minimal model* (usually called a minimal model) of S. Note that a minimal model is not necessarily unique for a given surface.

A *rational surface* is an irreducible smooth projective surface which is birational to $\mathbb{P}^2$. It is known that, for any surface S except rational surfaces, there is a unique relatively minimal model (the *minimal model* of S). For rational surfaces, the classification of relatively minimal models is known. Namely, there are two types of minimal models: the projective plane $\mathbb{P}^2$ and the Hirzebruch surfaces $\mathbb{F}_n$ ($n \geq 0$, $n \neq 1$) (cf. [1], Theorem V.10). Here a *Hirzebruch surface* $\mathbb{F}_n$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ associated with the sheaf $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$ for $n \geq 0$, where $\mathcal{O}_{\mathbb{P}^1}$ is the structure sheaf of $\mathbb{P}^1$ and $\mathcal{O}_{\mathbb{P}^1}(-n)$ is the inverse of the n tensor product of the Serre twisting sheaf $\mathcal{O}_{\mathbb{P}^1}(1)$.

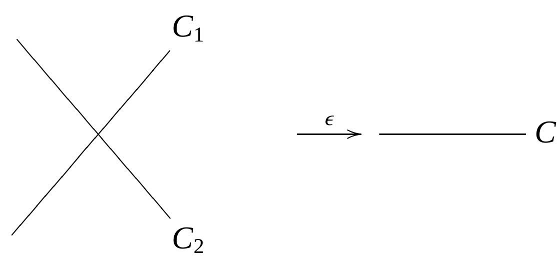
A *geometrically ruled surface* S over a curve C is an irreducible smooth projective surface together with a smooth morphism $\pi : S \to C$ such that all the fibers are isomorphic to C. Especially geometrically ruled surfaces over $\mathbb{P}^1$ are only Hirzebruch surfaces $\mathbb{F}_n$

(see [1], IV). The surfaces $\mathbb{F}_n$ are relatively minimal for any $n \geq 0$ except 1, and $\mathbb{F}_1$ is isomorphic to $\mathbb{P}^2$ blown up at one point.

3.2. **Proof of Lemma** 3.1**.** To show Lemma 3.1, here we compute the surface T_i obtained by blowing down all the (-1)-curves on $\tilde{S}_i$ appeared as the components of fibers of the morphism $\tilde{S}_i \to S_i \xrightarrow{\phi_i} \mathbb{P}^1$. Then we see that T_i is a geometrically ruled surface over $\mathbb{P}^1$. In Subsection 3.3 we show that T_i is a minimal model of $\tilde{S}_i$ for each i.

3.2.1. *Whitehead link case.* As we have seen in Section 2, $\phi_0 : S_0 \to \mathbb{P}^1$ has six degenerate fibers. Thus, the composite morphism of ϕ_0 and the blow-up morphism $\tilde{\phi}_0 : \tilde{S}_0 \to S_0 \xrightarrow{\phi_0} \mathbb{P}^1$ also has six degenerate fibers. Since the other fibers are conics in $\mathbb{P}^2$ they are isomorphic to $\mathbb{P}^1$. Here we show that we can blow down $\tilde{S}_0$ over $\mathbb{P}^1$ some number of times so that it becomes a geometrically ruled surface over $\mathbb{P}^1$.

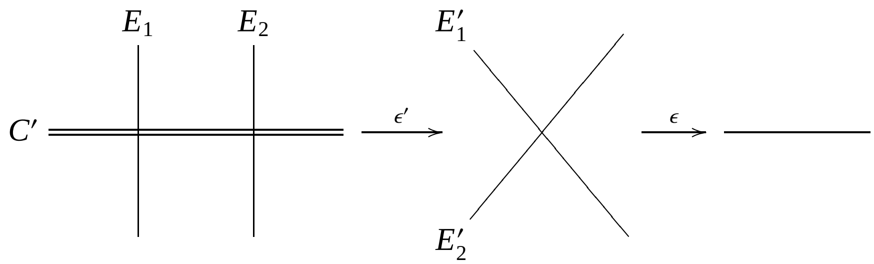
First consider the $\phi_0^{-1}(0 : 1)$ case. The fiber $\phi_0^{-1}(0 : 1)$ consists of two $\mathbb{P}^1$ which intersects each other transversally. There is no singular points of S_0 on this fiber. Hence it is isomorphic to $\tilde{\phi}_0^{-1}(0 : 1)$. Now we show that each curve C_i in the fiber has self-intersection number -1. By Lemma 3.4, $(C_1 + C_2) \cdot C_i = 0$. Since $(C_1 + C_2) \cdot C_i = C_i^2 + 1$, we have $C_i^2 = -1$. Thus we can blow down one of these two (-1)-curves in the fiber by Theorem 3.7, and the fiber becomes a curve $C \xrightarrow{\sim} \mathbb{P}^1$ with self-intersection number 0 by Proposition 3.6 (2), (3). We can work on the two fibers $\phi_0^{-1}(1 : \pm\frac{1}{\sqrt{2}})$ completely in the

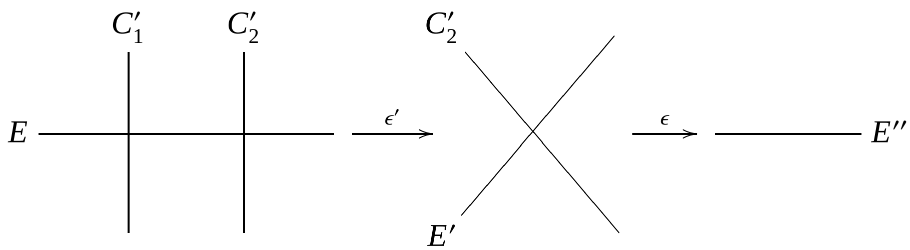


same way.

Next we consider the case $\phi_0^{-1}(1 : 0)$. As we have seen in Subsection 2.3, the fiber $\phi_0^{-1}(1 : 0)$ is a double $\mathbb{P}^1$ line C which contains two A_1 singular points of S_0. Then the divisor $\tilde{\phi}_0^*(1 : 0)$ in $\mathrm{Div}\,\tilde{S}_0$ is written as $2C' + E_1 + E_2$, where $\tilde{\phi}_0 : \tilde{S}_0 \to S_0 \xrightarrow{\phi_0} \mathbb{P}^1$, the curve C' is the strict transform of C in $\tilde{S}_0$ and E_i are the exceptional curves which are isomorphic to $\mathbb{P}^1$ with $E_i^2 = -2$ and $E_i \cdot C' = 1$. Since $\tilde{\phi}_0^*(1 : 0) \cdot C' = 0$, we have $C'^2 = -1$. Hence we can blow down at C' and the fiber becomes $E_1' + E_2'$ with $E'^2 = -1$ by Proposition 3.6 (2), (3). Thus we can blow down again and the fiber becomes $\mathbb{P}^1$.

Finally we consider the case $\phi_0^{-1}(1 : 1)$. We only have to consider the case $\phi_0^{-1}(1 : 1)$ (we can work on $\phi_0^{-1}(1 : -1)$ similarly). It consists of two rational curves C_1 and C_2 which intersects each other transversally at one point $(1 : 1 : 0,\ 1 : 1)$, which is an A_1 singular point of S_0. Hence the divisor $\tilde{\phi}_0^*(1 : 1)$ is $C_1' + C_2' + E$, where C_i' is the strict transform of C_i on $\tilde{S}_0$ and E is the exceptional curve obtained by resolving the singularity at $(1 : 1 : 0,\ 1 : 1)$. The exceptional curve E intersects with C_1', C_2' transversally at one point respectively and C_1' and C_2' do not meet at any point. Hence $E \cdot C_i' = 1$ and





$C_1' \cdot C_2' = 0$. Since $E^2 = -2$, we have $C_i'^2 = -1$. Thus we can blow down at C_1' and C_2' and obtain one rational curve E'' with self-intersection number 0.

Therefore the surface T_0 obtained by blowing down all the degenerate fibers of $\tilde{S}_0$ over $\mathbb{P}^1$ is a geometrically ruled surface over $\mathbb{P}^1$.

3.2.2. 6_2^2 *case.* As we have seen in Section 2, $\phi_1 : S_1 \to \mathbb{P}^1$ has eight degenerate fibers. Thus the composite morphism $\tilde{\phi}_1 : \tilde{S}_1 \to S_1 \xrightarrow{\phi_1} \mathbb{P}^1$ also has eight degenerate fibers. Here we show that the surface T_1 obtained by blowing down all the degenerate fibers of $\tilde{S}_1$ is also a geometrically ruled surface.

First note that, the four fibers $\phi_1^{-1}(1 : \frac{\sqrt{5}\pm 1}{2})$ and $\phi_1^{-1}(1 : -\frac{\sqrt{5}\pm 1}{2})$ are the unions of two $\mathbb{P}^1$s intersecting each other transversally at one point, and they do not contain any singular point of the surface S_1. Hence the situation is the same as the $\phi_0^{-1}(0 : 1)$ case. By the same manner as in the $\phi_0^{-1}(0 : 1)$ case, we can show that each curve in the four fibers have self-intersection number -1 and can be blown down to obtain one $\mathbb{P}^1$.

The fibers $\phi_1^{-1}(1 : 0)$ and $\phi_1^{-1}(1 : \pm 1)$ have the same shape and singular points as $\phi_0^{-1}(1 : 0)$ and $\phi_0^{-1}(1 : \pm 1)$ which appeared in the Whitehead link case. Therefore we see that the fibers $\tilde{\phi}_1^{-1}(1 : 0)$ and $\tilde{\phi}_1^{-1}(1 : \pm 1)$ can be blown down and we have $\mathbb{P}^1$. Note that the same argument as $\phi_1^{-1}(1 : 0)$ also works for $\phi_1^{-1}(0 : 1)$ since they are symmetric.

Hence the surface T_1 obtained by blowing down $\tilde{S}_1$ over $\mathbb{P}^1$ is a geometrically ruled surface over $\mathbb{P}^1$.

3.2.3. 6_3^2 *case.* As we have seen in the previous section, $\phi_2 : S_2 \to \mathbb{P}^1$ has four degenerate fibers. We can work on the fiber $\phi_2^{-1}(1 : 0)$ (resp. $\phi_2^{-1}(0 : 1)$) in the same way as in the $\phi_0^{-1}(1 : 0)$ (resp. $\phi_0^{-1}(0 : 1)$) case.

Now we consider the $\phi_2^{-1}(1 : 1)$ (resp. $\phi_2^{-1}(1 : -1)$) case. Since the point $(1 : 1 : 0,\ 1 : 1)$ (resp. $(1 : -1 : 0,\ 1 : -1)$) is an A_3 singular point, the exceptional curve consists of three rational curves E_1, E_2 and E_3 with $E_i^2 = -2$, $E_1 \cdot E_2 = E_3 \cdot E_2 = 1$ and $E_1 \cdot E_3 = 0$.

Then the divisor $\tilde{\phi}_2^*(1:1)$ (resp. $\tilde{\phi}_2^*(1:-1)$) is $2C' + E_1 + 2E_2 + E_3$ (see Figure 3), where C' is the strict transform on $\tilde{S}_2$ of the curve on S_2 defined by $x - y = 0$ (resp. $x + y = 0$).

Since $\tilde{\phi}_2^*(1:1)^2 = 0$ (resp. $\tilde{\phi}_2^*(1:\pm 1)^2 = 0$), we have $C'^2 = -1$. Hence we can also blow down this fiber for C', and the divisor of fiber on the blow down is $E_1 + 2E_2' + E_3$ with $E_2'^2 = -1$ and $E_1 \cdot E_2' = E_3 \cdot E_2' = 1$. Now the situation is the same as the $\tilde{\phi}_0^*(1:0)$ case. We can blow down the fiber twice to have one $\mathbb{P}^1$. Thus we obtain a geometrically ruled surface T_2 over $\mathbb{P}^1$ by blowing down $\tilde{S}_2$ repeatedly.

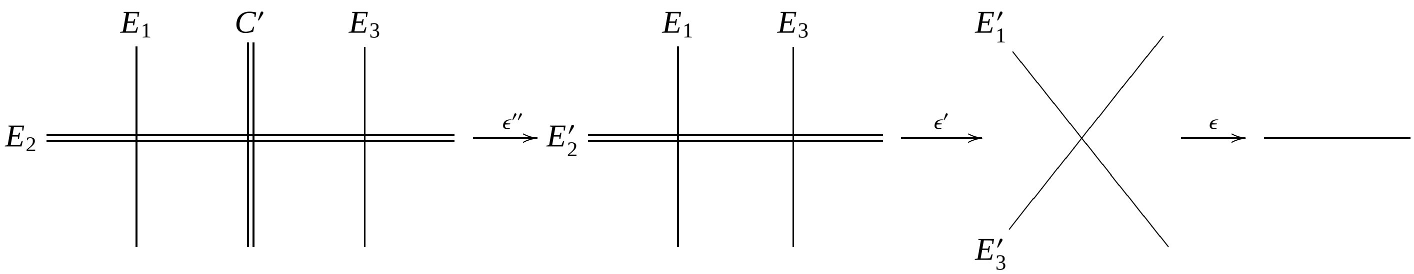


Figure 3. Blow down of $\tilde{\phi}_2^{-1}(1:\pm 1)$

3.3. **Proof of Lemma** 3.2**.** In Subsection 3.2 we have shown that the surfaces T_i obtained by blowing down the degenerate fibers of $\tilde{S}_i$ are geometrically ruled surfaces. Since Hirzebruch surfaces are the only rational surfaces which are geometrically ruled surfaces, each T_i is isomorphic to a Hirzebruch surface $\mathbb{F}_n$ for some $n \geq 0$. It remains to determine the number n. In fact we show that T_i are the Hirzebruch surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Therefore T_i are minimal models of $\tilde{S}_i$. Here we review a proposition on the Hirzebruch surfaces.

Proposition 3.8 (cf. [1], Prop. IV.1)**.** *Let $\mathbb{F}_n$ be a Hirzebruch surface ($n \geq 0$). Let $f \in \operatorname{Pic}\mathbb{F}_n$ be the element defined by a fiber of $\mathbb{F}_n$ over $\mathbb{P}^1$ and let $h \in \operatorname{Pic}\mathbb{F}_n$ be the element corresponding to the tautological line bundle (that is, the invertible sheaf $\mathcal{O}_{\mathbb{F}_n}(-1)$).*

(1) *$\operatorname{Pic}\mathbb{F}_n = \mathbb{Z}h \oplus \mathbb{Z}f$ with $f^2 = 0$ and $h^2 = n$.*
(2) *When $n > 0$, there exists a unique irreducible projective curve B on $\mathbb{F}_n$ with $b = [B]^2 < 0$. b is written as $b = h - nf$. Thus $b^2 = -n$.*
(3) *If $n \neq m$, two surfaces $\mathbb{F}_n$ and $\mathbb{F}_m$ are not isomorphic. $\mathbb{F}_n$ is a minimal model except if $n = 1$. $\mathbb{F}_0$ is $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$ is isomorphic to $\mathbb{P}^2$ with one point blown up.*

Note that we can take two sections $s_k : \mathbb{P}^1 \to S_i$ which are defined by

$$s_1 : (z:w) \mapsto (z:w:0,\ z:w),$$
$$s_2 : (z:w) \mapsto (w:z:0,\ z:w).$$

These two sections meet on S_i at the two points $(1:1:0,\ 1:1)$ and $(1:-1:0,\ 1:-1)$. We see that their lifts $\tilde{s}_k$ on the desingularization $\tilde{S}_i$ (that is, the strict transforms of $s_k(\mathbb{P}^1)$) do not intersect on the exceptional curves at $(1:1:0,\ 1:1)$ and $(1:-1:0,\ 1:-1)$, which means they do not intersect on $\tilde{S}_i$. Considering the process of blowing downs in Subsection 3.2 (see the figures of $\phi_i^{-1}(1:\pm 1)$), we know that their images in T_i also do

not intersect each other. Note that $s_1(\mathbb{P}^1) = V(xw - yz, u)$, $s_2(\mathbb{P}^1) = V(xz - yw, u)$ and

$$\begin{aligned} F_0 &= u^2 z(z^2 - 2w^2) + w(xz - yw)(xw - yz), \\ F_1 &= u^2(z^4 - 3z^2w^2 + w^4) + zw(xz - yw)(xw - yz), \\ F_2 &= u^2 z(z^2 - w^2) + w(xz - yw)(xw - yz). \end{aligned}$$

Then we can check that in $\operatorname{Div}(\tilde{S}_i)$

$$\operatorname{div}(xw - yz) = [V(xw - yz, u)], \quad \operatorname{div}(xz - yw) = [V(xz - yw, u)].$$

Thus we see that $\tilde{s}_k(\mathbb{P}^1)$ are linearly equivalent. Therefore we have

$$\tilde{s}_1(\mathbb{P}^1)^2 = \tilde{s}_2(\mathbb{P}^1)^2 = \tilde{s}_1(\mathbb{P}^1) \cdot \tilde{s}_2(\mathbb{P}^1) = 0.$$

The same is true for the images of $\tilde{s}_k(\mathbb{P}^1)$ in T_i.

Lemma 3.9. *Let C be an irreducible projective curve on a Hirzebruch surface $\mathbb{F}_n$ which is different from B. If $n > 0$, then $[C]^2 = 0$ if and only if C is a fiber of $\mathbb{F}_n$.*

Proof. We use the notation in Proposition 3.8. Put $b = [B]$ and $c = [C]$. Since $\operatorname{Pic}\mathbb{F}_n = \mathbb{Z}h \oplus \mathbb{Z}f$, there exist $\alpha, \beta \in \mathbb{Z}$ such that $c = \alpha h + \beta f$. First note that $c \cdot f \geq 0$ and $c \cdot b \geq 0$ since $B \neq C$. By Proposition 3.8, we know that $f^2 = 0$, $h \cdot f = 1$ and $b = h - nf$. Hence $\alpha \geq 0$ and $\beta \geq 0$. Since $c^2 = \alpha^2 n + 2\alpha\beta$, we see that $c^2 = 0$ if and only if $\alpha = 0$. If C is not a fiber on $\mathbb{F}_n$, then the restriction morphism $C \hookrightarrow \mathbb{F}_n \to \mathbb{P}^1$ is surjective. This means $c \cdot f > 0$. Therefore C is a fiber of $\mathbb{F}_n$. $\square$

Remember that the above sections $\tilde{s}_k$ are global sections. Hence we conclude that $n = 0$, namely the geometrically ruled surface T_i over $\mathbb{P}^1$ is $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, which is already a minimal model of $\tilde{S}_i$.

Remark 3.10. The set $S_i \smallsetminus V(p_i)$ of ideal points of the canonical component in $\mathbb{P}^2 \times \mathbb{P}^1$ consists of three 'ideal curves' for $i = 0, 2$, that is, one fiber $\phi_i^{-1}(1:0)$ and two global sections $s_k(\mathbb{P}^1)$. For S_1, there is an additional fiber. Namely $S_1 \smallsetminus V(p_1)$ consists of $\phi_1^{-1}(1:0)$, $\phi_1^{-1}(0:1)$ and two global sections $s_k(\mathbb{P}^1)$.

4. Desingularization of the models

In this section we determine $\tilde{S}_i$ in terms of the number of blow ups from their minimal models we have computed in Section 3.

From the result in Section 3, the smooth surfaces $\tilde{S}_i$ are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with one point blown up some number of times. Suppose that $\tilde{S}_i$ is obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by n one-point blow ups. Since $\chi(\mathbb{P}^1 \times \mathbb{P}^1) = 4$ we have $\chi(\tilde{S}_i) = \chi(\mathbb{P}^1 \times \mathbb{P}^1) + n = n + 4$ by repeatedly using Lemma 2.1. To determine the number n we have to compute $\chi(\tilde{S}_i)$. This is done by comparing the Euler characteristics of S_i and $\tilde{S}_i$. For the computation of $\chi(S_i)$ we follow the Landes' method in [7], §4. Here we introduce a rational map $\varphi : \mathbb{P}^2 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ defined by $(x : y : u,\ z : w) \mapsto (x : y,\ z : w)$ and set $\varphi_i := \varphi|_{S_i}$. This plays a crucial role for the computation of $\chi(S_i)$ in this section.

4.1. **Whitehead link case.** The computation of $\chi(S_0)$ has already been done in [7], §4. Here we review the computation for the completeness of this note. For more details, see op. cit.or the 6_2^2 case.

Let $\varphi_0 : S_0 \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the rational map defined by $(x : y : u,\ z : w) \mapsto (x : y,\ z : w)$. This is not defined at the three points $(0 : 0 : 1,\ 0 : 1)$ and $(0 : 0 : 1,\ 1 : \pm\frac{1}{\sqrt{2}})$. Let P_0 be the set of those three points and put $U_0 := S_0 \setminus P_0$. The image $\mathrm{Im}(\varphi_0)$ of U_0 is $\mathbb{P}^1 \times \mathbb{P}^1 \setminus Q_0$, where

$$
\begin{aligned}
Q_0 = {} & \mathbb{P}^1 \times \{(0:1)\} \setminus \{(1:0,\ 0:1),\ (0:1,\ 0:1)\} \\
& \sqcup \mathbb{P}^1 \times \left\{\left(1 : \tfrac{1}{\sqrt{2}}\right)\right\} \setminus \left\{\left(1 : \sqrt{2},\ 1 : \tfrac{1}{\sqrt{2}}\right),\ \left(1 : \tfrac{1}{\sqrt{2}},\ 1 : \tfrac{1}{\sqrt{2}}\right)\right\} \\
& \sqcup \mathbb{P}^1 \times \left\{\left(1 : -\tfrac{1}{\sqrt{2}}\right)\right\} \setminus \left\{\left(1 : -\sqrt{2},\ 1 : -\tfrac{1}{\sqrt{2}}\right),\ \left(1 : -\tfrac{1}{\sqrt{2}},\ 1 : -\tfrac{1}{\sqrt{2}}\right)\right\}.
\end{aligned}
$$

Hence $\chi(Q_0) = 0$. Let L_0 be the set of the above six points

$$
\begin{gathered}
(1:0,\ 0:1),\ (0:1,\ 0:1),\ \left(1 : \sqrt{2},\ 1 : \tfrac{1}{\sqrt{2}}\right), \\
\left(1 : \tfrac{1}{\sqrt{2}},\ 1 : \tfrac{1}{\sqrt{2}}\right),\ \left(1 : -\sqrt{2},\ 1 : \tfrac{-1}{\sqrt{2}}\right),\ \left(1 : \tfrac{-1}{\sqrt{2}},\ 1 : \tfrac{-1}{\sqrt{2}}\right).
\end{gathered}
$$

Let

$$F_0 = u^2z^3 - xyz^2w + (x^2 + y^2 - 2u^2)zw^2 - xyw^3 = G_0 + H_0u^2$$

be the decomposition of F_0 in terms of the power of u, where

$$G_0 = -xyz^2w + (x^2 + y^2)zw^2 - xyw^3, \quad H_0 = z(z^2 - 2w^2).$$

For $(z : w) \in \mathbb{P}^1$, we see that $H_0(z, w) = 0$ if and only if $(z : w) = (0 : 1),\ (1 : \pm\frac{1}{\sqrt{2}})$. Then we can check that the set $\{G_0 = H_0 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$ is equal to L_0. Therefore each point of L_0 has an infinite fiber isomorphic to the affine line $\mathbb{A}^1$. Hence we have $\chi(L_0) = 6$ and $\chi(\varphi_0^{-1}(L_0)) = 6$. Since $G_0 = w(xz - yw)(xw - yz)$, the set $B_0 := V(G_0) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is decomposed into the following three subsets

$$
\begin{aligned}
B_{01} &= V(w) = \mathbb{P}^1 \times \{(1:0)\} \subset \mathbb{P}^1 \times \mathbb{P}^1, \\
B_{02} &= V(xz - yw) = \{(1 : y,\ y : 1),\ (0:1,\ 1:0)\} \xrightarrow{\sim} \mathbb{P}^1, \\
B_{03} &= V(xw - yz) = \{(1 : y,\ 1 : y),\ (0:1,\ 0:1)\} \xrightarrow{\sim} \mathbb{P}^1.
\end{aligned}
$$

Their relations are as follows:

$$
\begin{aligned}
& B_{01} \cap B_{02} = \{(0:1,\ 1:0)\}, \quad B_{01} \cap B_{03} = \{(1:0,\ 1:0)\}, \\
& B_{02} \cap B_{03} = \{(1:1,\ 1:1),\ (1:-1,\ -1:1)\}, \quad B_{01} \cap B_{02} \cap B_{03} = \emptyset.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\chi(B_0) &= \chi(B_{01} \cup B_{02} \cup B_{03}) \\
&= \chi(B_{01} \cup B_{02}) + \chi(B_{03}) - \chi(B_{01} \cap B_{03}\ \cup\ B_{02} \cap B_{03}) \\
&= \chi(B_{01}) + \chi(B_{02}) + \chi(B_{03}) - \chi(B_{01} \cap B_{02}) - \chi(B_{01} \cap B_{03}) - \chi(B_{02} \cap B_{03}) + \chi(B_{01} \cap B_{02} \cap B_{03}) \\
&= 2 + 2 + 2 - 1 - 1 - 2 + 0 = 2.
\end{aligned}
$$

Thus we have

$$\begin{aligned}\chi(U_0) &= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1 \setminus (B_0 \sqcup Q_0)) + \chi(B_0 \setminus L_0) + \chi(\varphi_0^{-1}(L_0)) \\ &= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1) - \chi(B_0) - 2\chi(Q_0) - \chi(L_0) + \chi(\varphi_0^{-1}(L_0)) \\ &= 2 \times 4 - 2 - 6 + 6 = 6. \\ \chi(S_0) &= \chi(U_0) + \chi(P_0) = \chi(U_0) + 3 = 9.\end{aligned}$$

Thus $\chi(\tilde{S}_0) = \chi(S_0 \setminus S_{0,\mathrm{sing}}) + 4\chi(\mathbb{P}^1) = \chi(S_0) - 4 + 8 = 13$. Therefore $\tilde{S}_0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up $13 - 4 = 9$ times.

4.2. 6^2_2 **case.** Let $\varphi_1 : S_1 \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the rational map defined by $(x : y : u,\ z : w) \mapsto (x : y,\ z : w)$. This is not defined at the following four points

$$P_1 := \left\{\left(0 : 0 : 1,\ 1 : \pm\tfrac{1\pm\sqrt{5}}{2}\right)\right\}.$$

The image $\mathrm{Im}(\varphi_1)$ of the open subset $U_1 := S_1 \setminus P_1$ is $\mathbb{P}^1 \times \mathbb{P}^1 \setminus Q_1$, where

$$\begin{aligned}Q_1 = \ &\mathbb{P}^1 \times \left\{\left(1 : \tfrac{1+\sqrt{5}}{2}\right)\right\} \setminus \left\{\left(1 : \tfrac{\pm1+\sqrt{5}}{2},\ 1 : \tfrac{1+\sqrt{5}}{2}\right)\right\} \\ &\sqcup \mathbb{P}^1 \times \left\{\left(1 : \tfrac{1-\sqrt{5}}{2}\right)\right\} \setminus \left\{\left(1 : \tfrac{\pm1-\sqrt{5}}{2},\ 1 : \tfrac{1-\sqrt{5}}{2}\right)\right\} \\ &\sqcup \mathbb{P}^1 \times \left\{\left(1 : \tfrac{-1+\sqrt{5}}{2}\right)\right\} \setminus \left\{\left(1 : \tfrac{\pm1+\sqrt{5}}{2},\ 1 : \tfrac{-1+\sqrt{5}}{2}\right)\right\} \\ &\sqcup \mathbb{P}^1 \times \left\{\left(1 : \tfrac{-1-\sqrt{5}}{2}\right)\right\} \setminus \left\{\left(1 : \tfrac{\pm1-\sqrt{5}}{2},\ 1 : \tfrac{-1-\sqrt{5}}{2}\right)\right\}.\end{aligned}$$

Let L_1 be the subset of $\mathbb{P}^1 \times \mathbb{P}^1$ which consists of the eight points

$$\left(1 : \tfrac{\pm1+\sqrt{5}}{2},\ 1 : \tfrac{1+\sqrt{5}}{2}\right),\ \left(1 : \tfrac{\pm1-\sqrt{5}}{2},\ 1 : \tfrac{1-\sqrt{5}}{2}\right),\ \left(1 : \tfrac{\pm1+\sqrt{5}}{2},\ 1 : \tfrac{-1+\sqrt{5}}{2}\right),\ \left(1 : \tfrac{\pm1-\sqrt{5}}{2},\ 1 : \tfrac{-1-\sqrt{5}}{2}\right).$$

Let

$$F_1 := u^2z^4 - xyz^3w + (x^2 + y^2 - 3u^2)z^2w^2 - xyzw^3 + u^2w^4 = G_1 + H_1u^2$$

be the decomposition of F_1 in terms of the power of u, where

$$G_1 = -xyz^3w + (x^2 + y^2)z^2w^2 - xyzw^3, \quad H_1 = z^4 - 3z^2w^2 + w^4.$$

Then the image $\mathrm{Im}(\varphi_1)$ of φ_1 is decomposed into three subsets

$$\mathrm{Im}(\varphi_1) = \varphi_1(U_1) = \{G_1 = H_1 = 0\} \sqcup \{G_1 = 0,\ H_1 \neq 0\} \sqcup \{G_1 \neq 0,\ H_1 \neq 0\}.$$

We can characterize these three subsets as follows: For any point $(x : y,\ z : w) \in \mathrm{Im}(\varphi_1)$, the fiber of φ_1 at $(x : y,\ z : w)$ is an infinite set if and only if $G_1(x, y, z, w) = H_1(x, y, z, w) = 0$; the fiber of φ_1 at $(x : y,\ z : w)$ consists of one point if and only if $G_1(x, y, z, w) = 0$ and $H_1(x, y, z, w) \neq 0$; the fiber of φ_1 at $(x : y,\ z : w)$ consists of two points if and only if $G_1(x, y, z, w) \neq 0$ and $H_1(x, y, z, w) \neq 0$. For $(z : w) \in \mathbb{P}^1$, we see that $H_1(z, w) = 0$ if and only if $(z : w) = \left(1 : \pm\frac{1\pm\sqrt{5}}{2}\right)$. Then it is easy to see that L_1 is equal to the set of points satisfying $G_1(x, y, z, w) = H_1(x, y, z, w) = 0$. This means each point of L_1 has an infinite fiber which is isomorphic to the affine line $\mathbb{A}^1$. Hence $\chi(L_1) = 8$ and $\chi(\varphi_1^{-1}(L_1)) = 8$.

Since $G_1 = zw(xz - yw)(xw - yz)$, the set $B_1 := V(g_1) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is decomposed into 4 subsets

$$\begin{aligned}
B_{11} &= V(z) = \mathbb{P}^1 \times \{(0:1)\} \subset \mathbb{P}^1 \times \mathbb{P}^1,\\
B_{12} &= V(w) = \mathbb{P}^1 \times \{(1:0)\},\\
B_{13} &= V(xz - yw) = \{(1:y,\ y:1),\ (0:1,\ 1:0)\} \xrightarrow{\sim} \mathbb{P}^1,\\
B_{14} &= V(xw - yz) = \{(1:y,\ 1:y),\ (0:1,\ 0:1)\} \xrightarrow{\sim} \mathbb{P}^1.
\end{aligned}$$

Immediately we have

$$\begin{aligned}
&B_{11} \cap B_{12} = \emptyset, \quad B_{11} \cap B_{13} = \{(1:0,\ 0:1)\}, \quad B_{11} \cap B_{14} = \{(0:1,\ 0:1)\},\\
&B_{12} \cap B_{13} = \{(0:1,\ 1:0)\}, \quad B_{12} \cap B_{14} = \{(1:0,\ 1:0)\},\\
&B_{13} \cap B_{14} = \{(1:1,\ 1:1),\ (1:-1,\ -1:1)\},\\
&B_{11} \cap B_{12} \cap B_{13} = B_{11} \cap B_{12} \cap B_{14} = B_{11} \cap B_{13} \cap B_{14} = \emptyset,\\
&B_{12} \cap B_{13} \cap B_{14} = \emptyset, \quad B_{11} \cap B_{12} \cap B_{13} \cap B_{14} = \emptyset.
\end{aligned}$$

Hence we can compute the Euler characteristic $\chi(B_1)$:

$$\begin{aligned}
\chi(B_1) &= \chi(B_{11} \cup B_{12} \cup B_{13} \cup B_{14})\\
&= \chi(B_{11}) + \chi(B_{12} \cup B_{13} \cup B_{14}) - \chi((B_{11} \cap B_{12}) \ \cup\ (B_{11} \cap B_{13}) \ \cup\ (B_{11} \cap B_{14}))\\
&= \chi(B_{11}) + \chi(B_{12}) + \chi(B_{13}) + \chi(B_{14}) - \chi(B_{11} \cap B_{12}) - \chi(B_{11} \cap B_{13}) - \chi(B_{11} \cap B_{14})\\
&\quad - \chi(B_{12} \cap B_{13}) - \chi(B_{12} \cap B_{14}) - \chi(B_{13} \cap B_{14})\\
&= 2 + 2 + 2 + 2 - 0 - 1 - 1 - 1 - 1 - 2 = 2.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\chi(U_1) &= 2\chi((\mathbb{P}^1 \times \mathbb{P}^1) \setminus (B_1 \sqcup Q_1)) + \chi(B_1 \setminus L_1) + \chi(\varphi_1^{-1}(L_1))\\
&= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1) - \chi(B_1) - 2\chi(Q_1) - \chi(L_1) + \chi(\varphi_1^{-1}(L_1))\\
&= (2 \times 4) - 2 - (2 \times 0) - 8 + 8 = 6.\\
\chi(S_1) &= \chi(U_1) + \chi(P_1) = \chi(U_1) + 4 = 10.
\end{aligned}$$

Let $\tilde{S}_1$ be the desingularization of S_1 by blowing up at six singular points. Each fiber is a smooth conic curve inside $\tilde{S}_1$, which is isomorphic to $\mathbb{P}^1$. Thus we have

$$\chi(\tilde{S}_1) = \chi(S_1 \setminus S_{1,\mathrm{sing}}) + 6\chi(\mathbb{P}^1) = 10 - 6 + 12 = 16.$$

Remember that $\tilde{S}_1$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up n times. That means $\chi(\tilde{S}_1) = \chi(\mathbb{P}^1 \times \mathbb{P}^1) + n = n + 4$, which implies $n = 12$.

4.3. 6^2_3 **case.** We can compute $\chi(S_2)$ by the same way as in the 6^2_2 case, therefore we omit the details. Let $\varphi_2 : S_2 \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the rational map defined by $(x:y:u,\ z:w) \mapsto (x:y,\ z:w)$. This is not defined at three points $(0:0:1,\ 0:1)$, $(0:0:1,\ 1:1)$ and $(0:0:1,\ 1:-1)$. Let P_2 be the set of those three points and put

$U_2 := S_2 \setminus P_2$. The image $\mathrm{Im}(\varphi_2)$ of U_2 is $\mathbb{P}^1 \times \mathbb{P}^1 \setminus Q_2$, where

$$\begin{aligned} Q_2 &= \mathbb{P}^1 \times \{(0:1)\} \setminus \{(1:0,\ 0:1),\ (0:1,\ 0:1)\} \\ &\sqcup \mathbb{P}^1 \times \{(1:1)\} \setminus \{(1:1,\ 1:1)\} \\ &\sqcup \mathbb{P}^1 \times \{(1:-1)\} \setminus \{(1:-1,\ 1:-1)\}. \end{aligned}$$

Hence $\chi(Q_2) = 2$. Let

$$F_2 = u^2z^3 - xyz^2w + (x^2 + y^2 - u^2)zw^2 - xyw^3 = G_2 + H_2u^2$$

be the decomposition of F_2 in terms of the power of u, where

$$G_2 = -xyz^2w + (x^2 + y^2)zw^2 - xyw^3, \quad H_2 = z(z^2 - w^2).$$

For $(z : w) \in \mathbb{P}^1$, it is easy to check that $H_2(z, w) = 0$ if and only if $(z : w) = (0 : 1),\ (1 : 1)$ or $(1 : -1)$. Let L_2 be the subset of $\mathbb{P}^1 \times \mathbb{P}^1$ which consists of the following four points

$$(1:0,\ 0:1),\ (0:1,\ 0:1),\ (1:1,\ 1:1),\ (1:-1,\ 1:-1).$$

Then we see that $L_2 = \{G_2 = H_2 = 0\}$ as in the 6_2^2 case. Therefore each point of L_2 has an infinite fiber isomorphic to the affine line $\mathbb{A}^1$. Hence we have $\chi(L_2) = 4$ and $\chi(\varphi_2^{-1}(L_2)) = 4$. Since $G_2 = w(xz - yw)(xw - yz)$, the set $B_2 := V(g_2) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is decomposed into the following three subsets

$$\begin{aligned} B_{21} &= V(w) = \mathbb{P}^1 \times \{(1:0)\} \subset \mathbb{P}^1 \times \mathbb{P}^1, \\ B_{22} &= V(xz - yw) = \{(1:y,\ y:1),\ (0:1,\ 1:0)\} \xrightarrow{\sim} \mathbb{P}^1, \\ B_{23} &= V(xw - yz) = \{(1:y,\ 1:y),\ (0:1,\ 0:1)\} \xrightarrow{\sim} \mathbb{P}^1, \\ B_{21} \cap B_{22} &= \{(0:1,\ 1:0)\}, \quad B_{21} \cap B_{23} = \{(1:0,\ 1:0)\}, \\ B_{22} \cap B_{23} &= \{(1:1,\ 1:1),\ (1:-1,\ -1:1)\}, \quad B_{21} \cap B_{22} \cap B_{23} = \emptyset. \end{aligned}$$

Hence we have

$$\begin{aligned} \chi(B_2) &= \chi(B_{21} \cup B_{22} \cup B_{23}) \\ &= \chi(B_{21} \cup B_{22}) + \chi(B_{23}) - \chi(B_{21} \cap B_{23} \ \cup\ B_{22} \cap B_{23}) \\ &= \chi(B_{21}) + \chi(B_{22}) + \chi(B_{23}) - \chi(B_{21} \cap B_{22}) - \chi(B_{21} \cap B_{23}) - \chi(B_{22} \cap B_{23}) + \chi(B_{21} \cap B_{22} \cap B_{23}) \\ &= 2 + 2 + 2 - 1 - 1 - 2 + 0 = 2. \end{aligned}$$

Thus we can compute $\chi(S_2)$ as follows.

$$\begin{aligned} \chi(U_2) &= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1 \setminus (B_2 \sqcup Q_2)) + \chi(B_2 \setminus L_2) + \chi(\varphi_2^{-1}(L_2)) \\ &= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1) - \chi(B_2) - 2\chi(Q_2) - \chi(L_2) + \chi(\varphi_2^{-1}(L_2)) \\ &= 2 \times 4 - 2 - (2 \times 2) - 4 + 4 = 2. \\ \chi(S_2) &= \chi(U_2) + \chi(P_2) = \chi(U_2) + 3 = 5. \end{aligned}$$

We have already seen in Section 2 that $\chi(\tilde{S}_2) = \chi(S_2) + 8 = 5 + 8 = 13$. Hence $\tilde{S}_2$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up $13 - 4 = 9$ times.

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